Generalized Synchronization of Chaos

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What is Synchronization?

Figure: Starling Murmuration

Figure: Synchronous Fireflies

Complete Synchronization

Let's consider two diffusively coupled identical non-linear n-dimensional systems:

$$\dot{x_1} = f(x_1) + \alpha H(x_2 - x_1)$$
$$\dot{x_2} = f(x_2) + \alpha H(x_1 - x_2)$$
$$\Longrightarrow \dot{z} = \dot{x_1} - \dot{x_2} = f(x_1) - f(x_2) + 2\alpha Hz \quad \text{where} \quad z := x_1 - x_2$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a non-linear function and $H : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth coupling function. We investigate proper values of the coupling parameter α so that we could have $\lim_{t\to\infty} z(t) = 0$. By linearizing $\dot{z}$ near $x_1 = x_2$ we get:

$$\frac{dz}{dt} = [Df(x_1(t)) - 2\alpha I]z + O(\|Z\|^2)$$

where the first component is the First variational equation. Now we define $w(t) = e^{2\alpha t} z(t)$ and it adds an extra damping term $-\alpha z$ to $\dot{z}$ we get

$$\dot{w}(t) = [Df(x_1(t))]w$$

Suppose $\Phi(x_1(t))$ is the fundamental matrix for the solution of variational equation. Thus, we get

$$w(t) = \Phi(x_1(t))w(0)$$

Now we define Lyapunov exponent as:

$$\Lambda := \max_j \lim_{t\to\infty} \lambda_j(x_1(t))$$

where $\{\lambda_j(x_1(t))\}_{j=1}^n$ is the set of positive square roots of the symmetric matrix $\Phi(x_1(t))^T\Phi(x_1(t))$. The orbit $x_1(t)$ has Lyapunov Exponent Λ tells us that $\exists C > 0$ such that

$$\|e^{2\alpha t}z(t)\| = \|w(t)\| \leq C.e^{\Lambda t}$$
$$\implies \|z(t)\| \leq C.e^{(\Lambda-2\alpha)t}$$
$$\implies \alpha_c := \frac{\Lambda}{2}$$

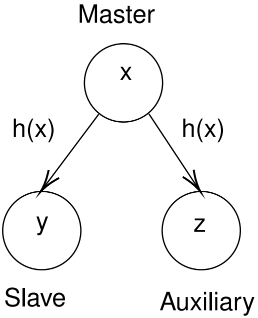
Generalized Synchronization

IS IT ALWAYS THIS MUCH EASY?
When the interacting systems are different. Generalized Synchronization can be observed by mapping it to a "CS" problem.
We will focus on the dynamics of unidirectional coupled systems. The master x and the slave y systems coupled as

$$\dot{x} = f(x)$$
$$\dot{y} = g(y, h(x))$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $h(x)$ is the coupling.

Auxiliary System Approach



To detect Generalized Synchronisation between the two systems x and y we add an auxiliary system z and investigate CS in between y and z. In case we observe CS between the identical copies, then GS is observed between the master and slave systems.

Rössler Driven by Lorenz

Master (Lorenz)

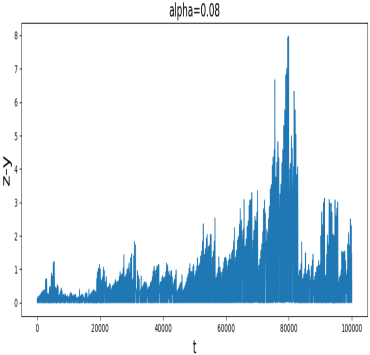
$$\dot{x}_1 = \sigma(x_2 - x_1),$$
$$\dot{x}_2 = x_1(\rho - x_3) - x_2,$$
$$\dot{x}_3 = -\beta x_3 + x_1 x_2$$

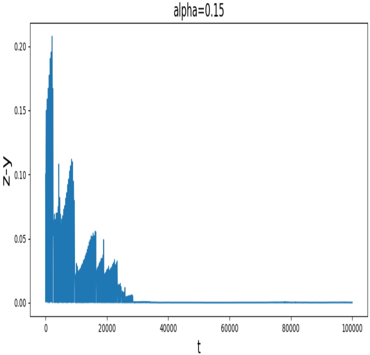
Slave (Rössler)

$$\dot{y}_1 = -(y_2 - y_3) + \alpha(x_1 - y_1),$$
$$\dot{y}_2 = y_1 + a y_2,$$
$$\dot{y}_3 = b + y_3(y_1 - c),$$

Auxiliary (Rössler)

$$\dot{z}_1 = -(z_2 - z_3) + \alpha(x_1 - z_1),$$
$$\dot{z}_2 = z_1 + a z_2,$$
$$\dot{z}_3 = b + z_3(z_1 - c),$$





where the parameters of Rössler system are $a = 0.2, b = 0.2, c = 5.7$ and the parameters of Lorenz system are $\rho = 28, \sigma = 10, \beta = 8/3$. Here the critical coupling is found to be $\alpha_c = 0.12$.

Later on

- Huge networks with different systems?
- Reconstruction techniques?

ANY QUESTİONS?

THANK YOU FOR LISTENING!