Seifert-Van Kampen Theorem

And Its Applications

Mentee: Mehmet Kırtışoğlu
Mentor: Feride Ceren Köse

DRP Turkey Symposium

August 27, 2022

Summary

What is Algebraic Topology?

Purpose

Algebra

- A set
- An operation
- Various properties
  - Closure
  - Associativity
  - Identity
  - Inverse
  - Comutativity

Topology

- A geometric object
- Continuous deformations without closing holes, opening holes, tearing, gluing, or passing through itself;
  - Streching
  - Twisting
  - Crumpling
  - Bending

$\Longrightarrow$ Algebraic Topology: Classification of topological spaces using the tools of abstract algebra

Purpose

Algebra

- A set
- An operation
- Various properties
  - Closure
  - Associativity
  - Identity
  - Inverse
  - Comutativity

Topology

- A geometric object
- Continuous deformations without closing holes, opening holes, tearing, gluing, or passing through itself;
  - Streching
  - Twisting
  - Crumpling
  - Bending

$\Longrightarrow$ Algebraic Topology: Classification of topological spaces using the tools of abstract algebra

Purpose

Algebra

- A set
- An operation
- Various properties
  - Closure
  - Associativity
  - Identity
  - Inverse
  - Comutativity

Topology

- A geometric object
- Continuous deformations without closing holes, opening holes, tearing, gluing, or passing through itself;
  - Streching
  - Twisting
  - Crumpling
  - Bending

$\Longrightarrow$ Algebraic Topology: Classification of topological spaces using the tools of abstract algebra

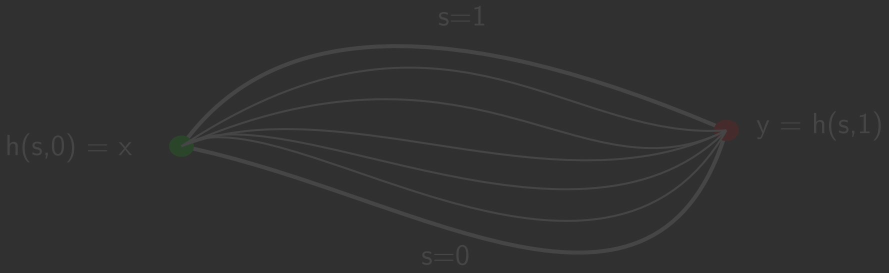
The Fundamental Group

Path Homotopy

Let X be a topological space and $x, y \in X$.

We say two paths $f, g : I \to X$ from x to y are **homotopic** if there exists a map, homotopy, $h : I \times I \to X$ such that

$$h(s,0) = x, \quad h(s,1) = y$$
$$h(0,t) = f(t), \quad h(1,t) = g(t)$$

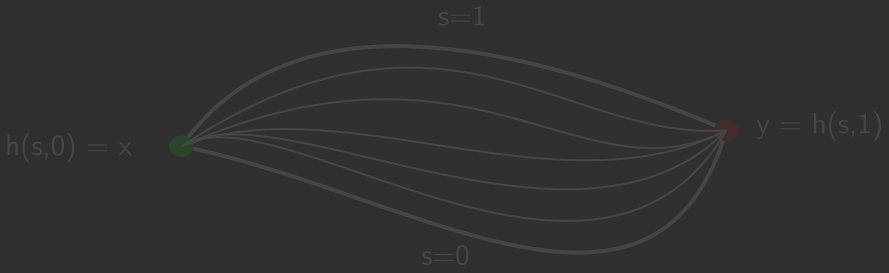


Path Homotopy

Let X be a topological space and $x, y \in X$.

We say two paths $f, g : I \to X$ from x to y are **homotopic** if there exists a map, homotopy, $h : I \times I \to X$ such that

$$h(s,0) = x, \quad h(s,1) = y$$
$$h(0,t) = f(t), \quad h(1,t) = g(t)$$

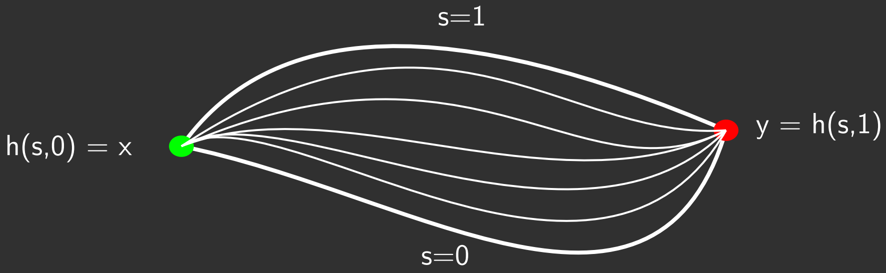


Path Homotopy

Let X be a topological space and $x, y \in X$.

We say two paths $f, g : I \to X$ from x to y are **homotopic** if there exists a map, homotopy, $h : I \times I \to X$ such that

$$h(s,0) = x, \quad h(s,1) = y$$
$$h(0,t) = f(t), \quad h(1,t) = g(t)$$



Definition of $\pi_1(X, x)$

Now we define $\pi_1(X, x)$ to be the set of equivalence classes of loops that start and end at x.

Then $\pi_1(X, x)$ becomes a group with the identity element $e = [c_x]$, the constant loop at x, and inverse of an element $[f]^{-1}$ is given by $[f^{-1}]$. It is also easy to check that composition of paths passes to equivalence classes via $[g][f] = [g.f]$.

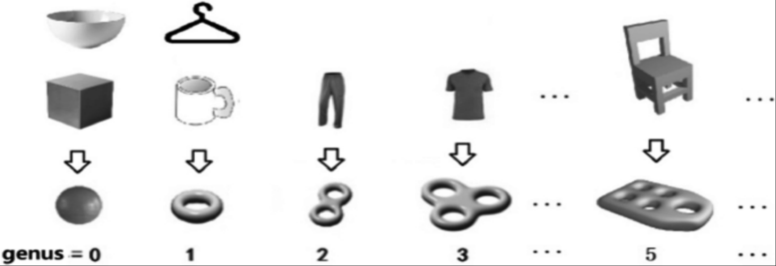


Figure: Genus [5]

Definition of $\pi_1(X, x)$

Now we define $\pi_1(X, x)$ to be the set of equivalence classes of loops that start and end at x.

Then $\pi_1(X, x)$ becomes a group with the identity element $e = [c_x]$, the constant loop at x, and inverse of an element $[f]^{-1}$ is given by $[f^{-1}]$. It is also easy to check that composition of paths passes to equivalence classes via $[g][f] = [g.f]$.

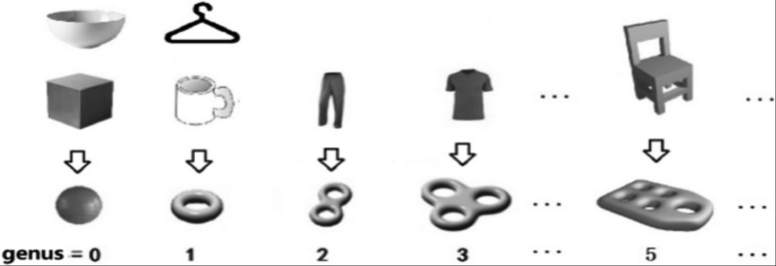
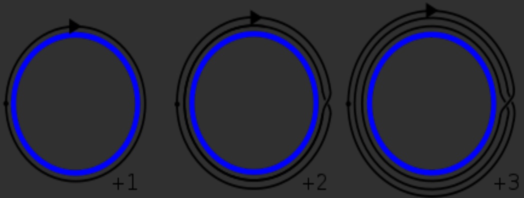
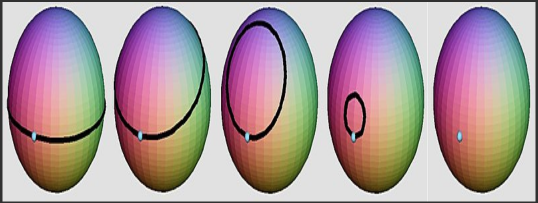


Figure: Genus [5]

Basic Examples

1. $\pi_1(\mathbb{R}, 0) = 0$
2. $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$
3. $\pi_1(\mathbb{S}^n, 1) = 0,\ n > 1$





Categories

Why Categories?

Algebraic Topology concerns mappings from topology to algebra.

And category theory provides us with a language to express these mappings.

Why Categories?

Algebraic Topology concerns mappings from topology to algebra.

And category theory provides us with a language to express these mappings.

What is Category?

A **category** $\mathfrak{C}$ consists of objects, a set $\mathfrak{C}(A, B)$ of morphisms between any two objects, an identity morphism id_A for each object A, and a composition law

$$\circ : \mathfrak{C}(B, C) \times \mathfrak{C}(A, B) \longrightarrow \mathfrak{C}(A, C)$$

for each triple of objects A, B, C where it must be associative[3].

A **functor** $F : \mathfrak{C} \to \mathfrak{D}$ is a map of categories where it assigns each object X in $\mathfrak{C}$ to an object $F(X)$ in $\mathfrak{D}$ and each morphism $f : X \to Y$ in $\mathfrak{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathfrak{D}$ such that

$F(id_X) = id_{F(X)}$, for all X in $\mathfrak{C}$

$F(g \circ f) = F(g) \circ F(f)$,

for all morphisms $f : X \to Y$, and $g : Y \to Z$ in C.

What is Category?

A **category** $\mathfrak{C}$ consists of objects, a set $\mathfrak{C}(A, B)$ of morphisms between any two objects, an identity morphism id_A for each object A, and a composition law

$$\circ : \mathfrak{C}(B, C) \times \mathfrak{C}(A, B) \longrightarrow \mathfrak{C}(A, C)$$

for each triple of objects A, B, C where it must be associative[3].

A **functor** $F : \mathfrak{C} \to \mathfrak{D}$ is a map of categories where it assigns each object X in $\mathfrak{C}$ to an object $F(X)$ in $\mathfrak{D}$ and each morphism $f : X \to Y$ in $\mathfrak{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathfrak{D}$ such that

$$F(id_X) = id_{F(X)}, \text{ for all } X \text{ in } \mathfrak{C}$$
$$F(g \circ f) = F(g) \circ F(f),$$

for all morphisms $f : X \to Y,$ and $g : Y \to Z$ in C.

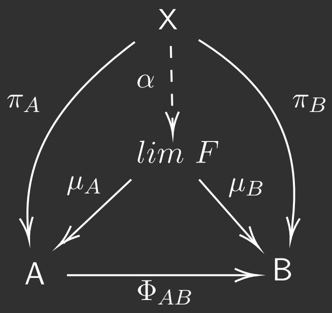
Examples

Category	Objects	Morphisms
Set	sets	functions
Top	topological spaces	continuous functions
Grp	groups	group homomorphisms
Ab	abelian groups	group homomorphisms
Vect$_K$	vector spaces over the field K	K-linear maps
Ring	rings	ring homomorphisms
Uni	uniform spaces	uniformly cts. functions
Top$_*$	pointed topological spaces	base preserving cts maps

Limits and Colimits

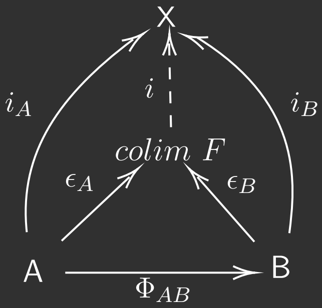
Definition of Limit

The **limit**[2] of a diagram $F : \mathfrak{I} \to \mathfrak{C}$ is an object $lim\ F$ in $\mathfrak{C}$ together with morphisms $\mu_A : lim\ F \to A$, for each A in the diagram, satisfying $\mu_B = \Phi_{AB} \circ \mu_A$ for every morphism $\Phi_{AB} : A \to B$ in the diagram. Moreover, these maps have the universal property. Diagramatically



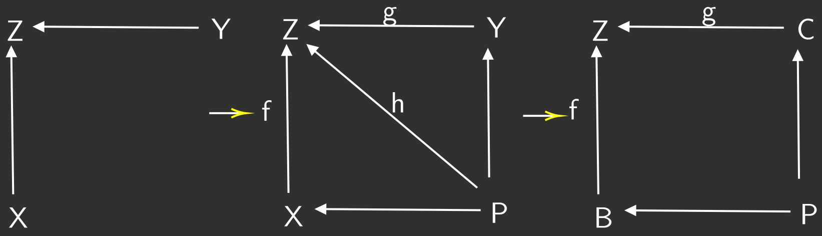
Definition of Colimit

The **colimit**[2] of a diagram $F : \mathfrak{I} \to \mathfrak{C}$ is an object $colim\ F$ in $\mathfrak{C}$ together with morphisms $\epsilon_A : A \to colim\ F$, for each A in the diagram, satisfying $\epsilon_A = \epsilon_B \circ \Phi_{AB}$ for every morphism $\Phi_{AB} : A \to B$ in the diagram. Moreover, these maps have the universal property. Diagramatically



Example of Limit

Example (Pullback): The limit P of the following diagram is called the $Pullback$ if it exists.

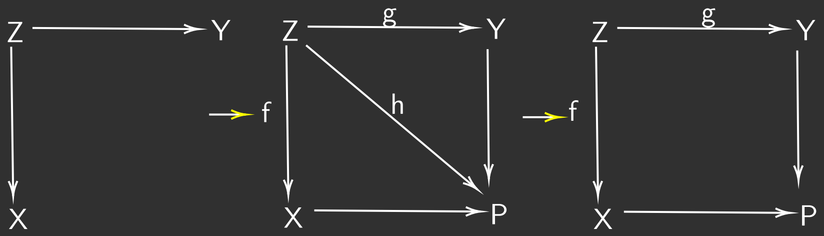


In the category of **Set**, the pullback exists and is given by the subset of $X \times Y$

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y.$$

Example of Colimit

Example (Pushout): The colimit P of the following diagram is called the $Pushout$ if it exists.



In the category of **Top**, the pushout exists and it is the quotient of the disjoint union $X \sqcup Y$ by $f(z) \sim g(z)$ for each $z \in Z$.

$$P = X \sqcup Y \Big/ f(z) \sim g(z)$$

Van Kampen With Two Subspaces

Statement

Suppose that $X = U \cup V$ where U, V, and $U \cap V \neq \emptyset$ are open and path connected. Then

$$\pi_1(X = U \cup V, x_0) \cong \pi(X_1, x_0) \sqcup_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$$

for any basepoint $x_0 \in U \cap V$[4].

Applications of the Van Kampen Theorem

Torus

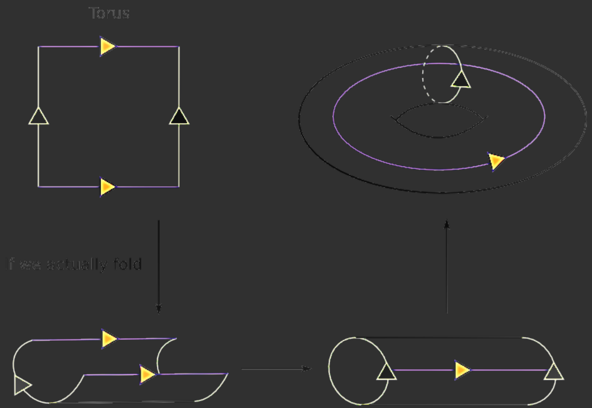
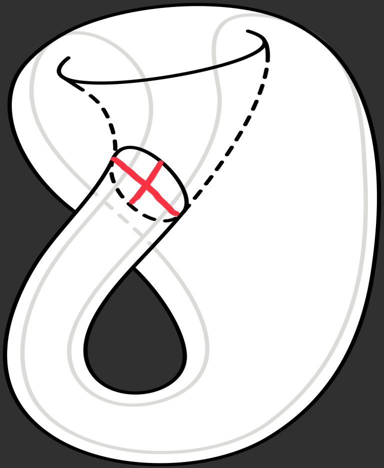


Figure: The Fundamental Polygon of Torus[1]

Klein Bottle



References

1. *Flat Life.* http://pi.math.cornell.edu/~mec/Winter2009/Victor/part1.htm.
2. *Limits and Colimits.* https://www.math3ma.com/blog/limits-and-colimits-part-2.
3. May, J. P. *A concise course in algebraic topology.* (University of Chicago press, 1999).
4. Møller, J. M. *The fundamental group and covering spaces.* 2011. https://arxiv.org/abs/1106.5650.
5. Zhang, T. in *Graphene* 83–103 (Springer, 2022).

Thanks!

Questions?